# Complete Integrability Prolongation Structure and Backlund Transformation for the Konno-Onno Equation

Chandan Kr. Das and A. Roy Chowdhury

High Energy Physics Division, Jadavpur University, Calcutta 700032, India

Reprint requests to Prof. A. R. Ch.; e-mail: arcphy@cal2.vsnl.net.in

Painleve analysis is used to study the complete integrability of the recently proposed Konno-Onno equation, which also leads to a general form of solutions of the system. An independent study, using the prolongation theory, gives the explicit form of the Lax pair which is then used to obtain the Backlund transformation connecting two sets of solutions of the system. The existence of the Lax pair and the positive result of the Painleve test indicate the complete integrability of the system.

*Key words:* Integrability; Backlund Transformation; Prolongation Structure, Lie Algebra.

## 1. Introduction

Analysis of the integrability of new nonlinear systems is one of the most important aspects of present day research [1]. One such new set of nonlinear equations was very recently suggested by Konno and Onno [2]. The set of equations under consideration is dispersionless and in many aspects similar to the three wave interaction process. It was also observed that under some constraint the equation is equivalent to the Sine-Gordon equation [3] used in particle physics. Incidentally it may be mentioned that the Konno-Onno equation was shown to be equivalent to the $O(3) \approx SU(2)$ nonlinear Sigma model of Pohlmeyer-Lund-Regge [4], again when some appropriate condition is imposed on the independent field variables. Due to these aspects the present set of equations is very important and deserves further study. Here we show that the Painleve test in the sense of Weiss, Tabor, and Carnavale [5] reveals that this set of nonlinear equations has all the properties for complete integrability. On the other hand, a prolongation structure analysis [6] yields the explicit form of the Lax pair, in which it is also possible to introduce a spectral parameter by invoking the invariance of the equation under a simple transformation. We also show that it is possible to deduce a Backlund transformation [7] from this Lax pair by utilising a transformation involving the prolongation variables.

## 2. Formulation

The nonlinear equation suggested by Konno and Onno can be written as

$$q_{xt} + (s\,r)_x = 0,$$
$$r_{xt} - 2\,q_x\,r = 0,$$
$$s_{xt} - 2\,q_x\,s = 0, \tag{1}$$

where $q$, $r$ and $s$ can be three physically relevant variables in a particular system. For example they can be a pump wave and two other density waves in a nonlinear plasma. To proceed with the Painleve test we set

$$q = \sum_{i=0}^{\infty} q_i\,\phi^{i+\alpha},$$
$$r = \sum_{i=0}^{\infty} r_i\,\phi^{i+\beta},$$
$$s = \sum_{i=0}^{\infty} s_i\,\phi^{i+\gamma}, \tag{2}$$

where $\phi = 0$ is the singular manifold and $\alpha$, $\beta$, $\gamma$ can be determined by matching the most singular terms. It turns out that $\alpha = \beta = \gamma = -1$. So the expansions are

$$q = \sum_{i=0}^{\infty} q_i\,\phi^{i-1},$$
$$r = \sum_{i=0}^{\infty} r_i\,\phi^{i-1},$$
$$s = \sum_{i=0}^{\infty} s_i\,\phi^{i-1}, \tag{3}$$

where the leading coefficients are given as

$$q_0 = -\phi_t$$

and

$$r_0\,s_0 = -\phi_t^2. \tag{4}$$

Substituting (2) in (1) we get

$$(i-1)(i-2)\,\phi_t\,q_i+(i-2)\,s_0\,r_i+(i-2)\,r_0\,s_i$$
$$= \text{terms involving } (r_{i-1}, s_{i-1}, q_{i-1}),$$

$$-2\,(i-1)\,r_0\,q_i+i\,(i-3)\,\phi_t\,r_i$$
$$= \text{terms involving } (r_{i-1}, s_{i-1}, q_{i-1}),$$

$$-2\,(i-1)\,s_0\,q_i+i\,(i-3)\,\phi_t\,s_i$$
$$= \text{terms involving } (r_{i-1}, s_{i-1}, q_{i-1}).$$

$$(5)$$

So resonances are determined from (5) by

$$\det \begin{vmatrix} (i-1)(i-2)\,\phi_t & (i-2)\,s_0 & (i-2)\,r_0 \\ -2\,(i-1)\,r_0 & i\,(i-3)\,\phi_t & 0 \\ -2\,(i-1)\,s_0 & 0 & i\,(i-3)\,\phi_t \end{vmatrix} = 0, \quad (6)$$

which simplyfies to

$$i\,(i-1)\,(i-2)\,(i-3)\,(i^2-3\,i-4)\,\phi_t^3 = 0. \quad (7)$$

So the resonances are at

$$i = -1, 0, 1, 2, 3, 4. \quad (8)$$

Since we have three second order equations, the existence of six resonances is the first criterion for the Painleve test. The next important aspect is the existence of a sufficient number of arbitrary constants and compatibility with the truncation of the series (3). So, if we truncate at the zeroth level, we get

$$q = \frac{q_0}{\phi} + q_1,$$

$$r = \frac{r_0}{\phi} + r_1,$$

$$s = \frac{s_0}{\phi} + s_1, \quad (9)$$

and the arising conditions are

1)    $q_0 = -\phi_t$,                           (10a)

2)    $r_0\,s_0 = -\phi_t^2$,                     (10b)

3)    $r_0\,s_1 + r_1\,s_0 = \phi_{tt}$,          (10c)

4)    $r_{0x}\,\phi_t + r_{0t}\,\phi_x - r_0\,\phi_{xt} + 2\,r_1\,\phi_x\,\phi_t = 0$,    (10d)

5)    $s_{0x}\,\phi_t + s_{0t}\,\phi_x - s_0\,\phi_{xt} + 2\,s_1\,\phi_x\,\phi_t = 0$,    (10e)

6)    $r_{0xt} + 2\,\phi_{xt}\,r_1 - 2\,r_0\,q_{1x} = 0$,    (10f)

7)    $s_{0xt} + 2\,\phi_{xt}\,s_1 - 2\,s_0\,q_{1x} = 0$,    (10g)

along with the stipulation that $(q_1, r_1, s_1)$ is another set of solutions of the same equation, so that one can consider (9) as a Backlund transformation. Though it is possible to proceed with these equations in general, we assume a special realisation of $(s_0, r_0)$ as $r_0 = \phi_t$, $s_0 = -\phi_t$, whence from (10d, e) we get

$$r_1 = -\frac{\phi_{tt}}{2\,\phi_t}; \quad s_1 = \frac{\phi_{tt}}{2\,\phi_t}. \quad (11)$$

Substituting in (10f, g) we get

$$[\phi, t]_x = 0, \quad (12)$$

where $[\phi, t]$ stands for the Schwarzian derivative with respect to $t$,

$$[\phi, t] = \frac{\partial}{\partial_t}\left(\frac{\phi_{tt}}{\phi_t}\right) - \frac{\phi_{tt}^2}{\phi_t^2}, \quad (13)$$

and finally one gets

$$q = -\frac{\phi_t}{\phi} + \frac{\phi_{tt}}{2\,\phi_t} + f_1\,(t),$$

$$r = \frac{\phi_t}{\phi} - \frac{\phi_{tt}}{2\,\phi_t},$$

$$s = -\frac{\phi_t}{\phi} + \frac{\phi_{tt}}{2\,\phi_t}. \quad (14)$$

Obeing an arbitrary function of $(x, t)$ with the only condition that it is a solution of (12), eq. (14) gives a solution of the coupled system (1). It is actually possible to verify that (14) satisfies (1) identically and $f_1\,(t)$ is an arbitrary integration constant. It may be noted that one way of solving (12) is to assume

$$[\phi, t] = 0; \quad (15)$$

whose solution can be written as

$$\phi = f\left(\frac{a\,(x)\,t + b\,(x)}{c\,(x)\,t + d\,(x)}\right) \quad (16)$$

with $a, b, c, d$ being arbitrary functions of $x$, and $\phi$ satisfying

$$\frac{\partial}{\partial_z}\left(\frac{f_{zz}}{f_z}\right) - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2 = 0, \quad (17)$$

whose solution is $f = z$. On the other hand, if we assume

$$[\phi, t] = \lambda\,(t), \quad (18)$$

then by substituting $\psi' = 1/(\phi_t)^2$ one can convert (18) into the Schrödinger equation

$$\psi'_{tt} + 2\,(\lambda\,(t) - 6\,r_1^2)\,\psi' = 0, \quad (19)$$

which is a linear equation in the variable "$t$". But the actual determination of the Lax pair through the Painleve analysis becomes a difficult problem. So we take recourse of the prolongation theory, being encouraged by the fact that all the conditions of the Painleve test are satisfied.

## 3. Prolongation Analysis

To proceed with prolongation analysis we define new independent variables $l = q_x$, $m = r_x$, $n = s_x$, so that the set (1) can be written as a collection of differentials of two forms:

$$\alpha_1 = dq \wedge dt - l\, dx \wedge dt\,,$$
$$\alpha_2 = dr \wedge dt - m\, dx \wedge dt\,,$$
$$\alpha_3 = ds \wedge dt - n\, dx \wedge dt\,. \tag{20}$$
$$\alpha_4 = -dl \wedge dx + (m\,s + n\,r)\, dx \wedge dt\,,$$
$$\alpha_5 = -dm \wedge dx - 2\, l\,r\, dx \wedge dt\,,$$
$$\alpha_6 = -dn \wedge dx - 2\, l\,s\, dx \wedge dt\,. \tag{21}$$

A simple computation shows that these differentials are closed under exterior derivative, that is

$$d\alpha_i = \sum_{j=1}^{6} \sigma_j^i\, \alpha_j\,. \tag{22}$$

It is then important to search for a differential form,

$$w_k = dy^k + F^k\, dx + G^k\, dt\,, \tag{23}$$

where $y^k$ are prolongation variables, $F^k$, $G^k$ depend on $(q, r, s, l, m, n, y)$ and $w_k$ satisfies

$$dw_k = \sum_{j=1}^{6} \beta_j^k\, \alpha_j + \sum_j (\alpha_j^k\, dx + b_j^k\, dt) \wedge w_j\,, \tag{24}$$

which is nothing but the generalised closure condition. A simple calculation yields

$$F = l\, X_1 + m\, X_2 + n\, X_3\,,$$
$$G = r\, X_6 + s\, X_7 + X_1\,, \tag{25}$$

with the commutation relations

$$[X_1, X_2] = -X_6,\ [X_1, X_7] = -2\, X_3\,,$$
$$[X_1, X_3] = -X_7,\ 2\,[X_2, X_3] + [X_6, X_7] = 0\,,$$
$$[X_1, X_6] = -2\, X_2,\ [X_2, X_6] = 0\,,$$
$$[X_2, X_7] = X_1 = [X_3, X_6],\ [X_3, X_7] = 0\,. \tag{26}$$

One may note that $(X_1, X_2, X_6)$ and $(X_1, X_3, X_7)$ form a closed sub-algebra. We also can set $X_6 = \delta X_2$, whence $X_7 = -\delta X_3$, so that

$$F = l\, X_1 + m\, X_2 + n\, X_3\,,$$
$$G = \delta r\, X_2 - \delta s\, X_3 + 1/2\, \delta^2\, X_1 \tag{27}$$

with $[X_1, X_2] = -2/\beta\, X_2\ [X_2, X_3] = -1/\beta\, X_1\ [X_1, X_3] = 2/\beta\, X_3$. Using the adjoint representation of this Lie algebra

we get the following form of two Lax equations:

$$\psi_x = F\psi;\ \psi_t = G\psi\,,$$

$$F = \frac{1}{\delta} \begin{pmatrix} 0 & 2s_x & -r_x \\ 2r_x & -2q_x & 0 \\ -2s_x & 0 & 2q_x \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & -s & -r \\ 2r & -\delta & 0 \\ 2s & 0 & \delta \end{pmatrix}. \tag{28}$$

It is interesting to observe that another form of $F$, $G$ can be

$$F = l\, X_1 + m\, X_2 + n\, X_3 + X_4\,,$$
$$G = q\, X_5 + r\, X_6 + s\, X_7 + X_8 + r\,s\, X_9\,. \tag{29}$$

The commutation rules so obtained allow the reductions

$$X_6 = \delta X_2,\ X_7 = -\delta X_3,\ X_8 = 1/2\, \delta^2\, X_1,\ X_9 = 1/\alpha\, X_4\,,$$

so that,

$$F = l\, X_1 + m\, X_2 + n\, X_3 + X_4\,,$$
$$G = \delta r\, X_2 - \delta s\, X_3 + \frac{1}{2}\, \delta^2\, X_1 + \frac{r\,s}{d}\, X_4\,, \tag{30}$$

along with the commutation rules

$$[X_1, X_4] = 0,\ i = 1, 2, 3\,,$$
$$[X_1, X_2] = -\frac{2}{\delta}\, X_2,\ [X_1, X_3] = \frac{2}{\delta}\, X_3\,,$$
$$[X_2, X_3] = -\frac{1}{\delta}\, X_1 - \frac{1}{\delta\, d}\, X_4\,. \tag{31}$$

## 4. Riccati Equation

Let us consider (28) for the eigenvector $\psi = (y_1, y_2, y_3)^t$ (with $\delta = 1$), which yields

$$y_{1x} = s_x\, y_2 - r_x\, y_3\,,$$
$$y_{2x} = 2\, r_x\, y_1 - 2\, q_x\, y_2\,,$$
$$y_{3x} = -2\, s_x\, y_1 - 2\, q_x\, y_3\,. \tag{32}$$

Let us set $\phi_1 = y_1/y_2$, $\phi_2 = y_3/y_2$, which yields

$$\phi_{1x} = s_x - r_x\, \phi_2 - 2\, r_x\, \phi_1^2 + 2\, q_x\, \phi_1\,,$$
$$\phi_{2x} = -2\, s_x\, \phi_1 + 4\, q_x\, \phi_2 - 2\, r_x\, \phi_1\, \phi_2\,, \tag{33}$$

We next observe that the transformation

$$\phi_1 \rightarrow \phi_1'\,,$$
$$\phi_2 \rightarrow \phi_2'\,,$$
$$r_x \rightarrow -r_x' + \phi_1'\,,$$

$$s_x \rightarrow \phi_1' \ \phi_2' - s_x',$$

$$q_x \rightarrow -\phi_1'^2 + q_x' \tag{34}$$

keeps the Riccati equation (33) form-invariant, that is $(\phi_1', \phi_2')$ also satisfies

$$\phi_{1,x}' - s_x' - r_x' \ \phi_2' - 2 \ r_x' \ \phi'^2 + 2 q_x' \ \phi_1',$$

$$\phi_{2,x}' = -2 \ s_x' \ \phi_1' + 4 \ q_x' \ \phi_2' - 2 \ r_x' \ \phi_1' \ \phi_2' \ . \tag{35}$$

So, when (34) is combined with the time part of the Riccati equations

$$\phi_{1t} = -s \ r \ \phi_2 - 2 \ r \ \phi_2^2 + \phi_1,$$

$$\phi_{2t} = 2 \ s \ \phi_1 - 2 \ r \ \phi_2 \ \phi_1 + 2 \ \phi_2 \tag{36}$$

we get relations between the two sets of solutions $(q, r, s)$ and $(q', r', s')$, written as follows

$$q_x' - q_x = (r_x + r_x')^2,$$

$$\left( \frac{s_x + s_x'}{r_x + r_x'} \right)_x = 2 s_x (r_x + r_x') + 4 q_x \left( \frac{s_x + s_x'}{r_x + r_x'} \right)$$

$$+ 2 r_x (s_x + s_x'),$$

$$r_{xt} + r_{xt}' = s + r \left( \frac{s_x + s_x'}{r_x + r_x'} \right) + 2 r (r_x + r_x')^2 + (r_x + r_x'),$$

$$\left( \frac{s_x + s_x'}{r_x + r_x'} \right)_t = -2 s (r_x + r_x') + 2 \frac{s_x + s_x'}{r_x + r_x'} + 2 r (s_x + s_x'). \tag{37}$$

Therefore these relations can be denoted as the Backlund transformation. Note that there is no parameter in our Lax pair and hence in (37). To introduce a parameter, we observe that the transformation

$$x' = \lambda x,$$

$$t' = t,$$

$$q = q + \lambda,$$

$$r = \lambda r,$$

$$s = \frac{1}{\lambda} s \tag{38}$$

keeps the equation invariant. So, if we impose this invariance to operate on the one form $w_k$, then the form of the matrices $F$, $G$ turns out to be

$$F = \begin{pmatrix} 0 & \lambda s_x & -\frac{1}{\lambda} r_x \\ 2 \frac{1}{\lambda} r_x & -2 q_x & 0 \\ -2 \lambda s_x & 0 & 2 q_x \end{pmatrix}, \tag{39}$$

$$G = \begin{pmatrix} 0 & \lambda s & -\frac{1}{\lambda} r \\ 2 \frac{1}{\lambda} r & -1 & 0 \\ -2 \lambda s & 0 & 1 \end{pmatrix}, \tag{40}$$

where we have set $\beta = 1$. The same parameter $\lambda$ will also occur in the Backlund transformation if one now does the same calculation as before with these $F$ and $G$.

## 5. Explicit Solutions

After proving the complete integrability of the system we now search for explicit solutions of the system given by (1). We first assume the $q = f(x - v \ t)$, $r = g(x - v \ t)$, and $s = h(x - v \ t)$, with $x - v \ t = \xi$. It is easily observed that one gets

$$f_\xi^2 + g_\xi h_\xi = L \text{ (constant)} \tag{41}$$

along with

$$(h_\xi g - h g_\xi)_\xi = 0 \tag{42a}$$

and

$$f_\xi \ g = -\frac{1}{2} \ v \ g_{\xi\xi} . \tag{42b}$$

From (42a) we at once get

$$h = N \ g.$$

$N$ is an integration constant, so that

$$f_\xi^2 = L + N \ g_\xi^2. \tag{43}$$

Combining with (41), we get

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = \left[ \frac{L}{N} + \frac{N}{v^2} \ (g^2 - Q)^2 \right]^{1/2} \tag{44}$$

or

$$\xi = \int^g \frac{\mathrm{d}g}{(\alpha \ g^4 + \beta \ g^2 + \gamma)^{1/2}} , \tag{45}$$

where

$$\alpha = \frac{N}{v^2} ,$$

$$\beta = -\frac{2 \ N \ Q}{v^2} ,$$

$$\gamma = \frac{L}{N} - \frac{N \ Q^2}{v^2} , \tag{46}$$

and $Q$ is some arbitrary constant.

This integral can be evaluated by noting that we can write it as

$$\xi = \frac{1}{\sqrt{\alpha}} \int^g \frac{dg}{\sqrt{(g^2 + a)(g^2 + b)}} \qquad (47)$$

with

$$ab = \frac{\gamma}{\alpha}, \quad a + b = \frac{\beta}{\alpha}.$$

Now it is known that

$$a \int^g \frac{dg}{\sqrt{(g^2 | a)(g^2 | b)}} = Sc^{-1}\left( \frac{g}{b} \middle| \frac{a^2 - b^2}{a^2} \right),$$

$Sc (g/b | \sigma)$ being an elliptic function. So we find

$$a \sqrt{\alpha} \cdot \xi = Sc^{-1}\left( \frac{g}{b} \middle| \frac{a^2 - b^2}{a^2} \right)$$

or

$$g = b\, Sc\left( a \sqrt{\alpha}\, \xi \middle| \frac{a^2 - b^2}{a^2} \right). \qquad (48)$$

Hence $h$ is also known, but the quadrature needed for the evaluation of $f$ can not be done analytically, so this actually represents a propagating solution of the system. If $r = 0$, we get

$$\xi = \int \frac{dg}{g \sqrt{\alpha \cdot g^2 + \beta}}, \qquad (49)$$

which yields

$$g = -\sqrt{\frac{\beta}{\alpha}}\ \mathrm{Cosech}\,(\xi \sqrt{\beta}) \qquad (50)$$

and $h = -N \sqrt{\beta/\alpha}\ \mathrm{Cosech}\,(\xi \sqrt{\beta})$, but again for the evaluation of $f$ we observe

$$f = \int d\xi \sqrt{L + 2N \frac{\beta^{3/2}}{\alpha} \mathrm{Cosech}^2(\xi \sqrt{\beta})\, \mathrm{Coth}^2(\xi \sqrt{\beta})},$$

which can be done only numerically. But if $L = 0$, then it can be obtained at once. Besides the propagating solution we can also obtain a rational solution from the Painleve analysis results given above. It is easy to verify that

the expressions of $(q, s, r)$ given in (14) identically satisfies the equation set (1). So, if we can find $\phi$ we have another set of solutions. Now from (18) we get

$$m_t - \frac{1}{2} m^2 - \sigma(t), \qquad (51)$$

where $m = (\ln \phi_t)_t$. But (51) being a general Riccati equation, can not be solved in totality. But if the integration constant $\sigma$ is chosen to be zero we get immediately, after simple integration,

$$q = -\frac{C_3(x)}{C_3(x)(t + C_1(x)) - K(x)} = -r = s,$$

where $C_1$, $C_3$ and $K$ are arbitrary functions of $x$. One should note that this solution can be singular if $C_3(x)(t + C_1(x)) - K(x) = 0$. The same is also true for the Cosech type solution given in (50).

## 6. Conclusion

In our analysis we have shown that complete integrability of the Konno-Onno equations can be established both in the Lax sense and in the Painleve sense. The Painleve analysis leads to some special rational solutions. The Lax pair obtained is used to deduce the Backlund Transformation.

[1] Integrability of Nonlinear Systems. Ed.: Koshzmann Schwarzbach et al. (LNP495) Springer Verlag, Berlin 1996.
[2] K. Konno and H. Onno, J. Phys. Soc. Japan **65**, 340 (1996).
[3] R. Hirota and S. Tsugimoto, J. Phys. Soc. Japan **63**, 3533 (1994); and K. Konno and H. Onno, J. Phys. Soc. Japan **63**, 3534 (1994).
[4] V. P. Kotlyarov, J. Phys. Soc. Japan **63**, 3535 (1994).
[5] J. Weiss, M. Tabor, and Gl. Carnavale, J. Math. Phys. **24**, 522 (1983).
[6] A. Roy, Chowdhury and T. Roy, J. Math. Phys. **21,** 189 (1980); and H. D. Whalquist and F. B. Estabrook, J. Math. Phys. **16,** 1 (1975).
[7] Backlund Transformation. – Lecture Notes in Physics 515, Ed. by: Dodd and Eckmann Springer Verlag, Berlin 1976.